TOWARDS SAMPLE PATH ESTIMATES FOR FAST-SLOW STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

MANUEL V. GNANN, CHRISTIAN KUEHN, AND ANNE PEIN

ABSTRACT. Estimates for sample paths of fast-slow stochastic ordinary differential equations (SODEs) have become a key mathematical tool relevant for theory and applications. In particular, there have been breakthroughs by Berglund and Gentz to prove sharp exponential error estimates. In this paper, we take the first steps in order to generalize this theory to fast-slow stochastic partial differential equations (SPDEs). In a simplified setting with a natural decomposition into low- and high-frequency modes, we demonstrate that for a short time period the probability for the corresponding sample path to leave a neighbourhood around the stable slow manifold of the system is exponentially small as well.

1. INTRODUCTION

A general fast-slow SODE system has the form

$$\begin{cases} \mathrm{d}u &= \frac{1}{\varepsilon} f(u,v,\varepsilon)\ \mathrm{d}s + \frac{\sigma_f}{\sqrt{\varepsilon}} F(u,v,\varepsilon)\ \mathrm{d}W, \\ \mathrm{d}v &= g(u,v,\varepsilon)\ \mathrm{d}s + \sigma_g G(u,v,\varepsilon)\ \mathrm{d}W, \end{cases} \tag{1.1}$$

where $(u,v) = (u(s),v(s)) \in \mathbb{R}^{m+n}$, $0 < \varepsilon, \sigma_f, \sigma_g \ll 1$ are small parameters, $W = W(s)$ is a k-dimensional vector of independent identically distributed (iid) Brownian motions, and the maps f, g, F, G are assumed to be sufficiently smooth. Furthermore, all maps have suitable domains and ranges, e.g., $F : \mathbb{R}^{m+n} \times \mathbb{R} \to \mathbb{R}^{m\times k}$ is matrix-valued. The parameter ε controls the time scale separation between the fast u variables and the slow v variables, while the parameters σ_f and σ_g regulate the noise level. SODEs of the form (1.1) appear in many modelling contexts. Examples are neuroscience [21, 26], climate science [1, 22], and ecology [24, 25], among many other areas. Without noise ($\sigma_f = 0 = \sigma_g$), fast-slow ODEs have appeared in virtually all branches of science and engineering; we refer to [17] for a detailed overview and references.

Within this paper, we focus on the case where the critical set of the deterministic system, i.e.

$$\mathcal{C}_0 := \{(u,v) \in \mathbb{R}^{m+n} : f(u,v,0) = 0\}, \tag{1.2}$$

is a normally hyperbolic attracting manifold. Thus by Fenichel's Theorem [11, 16, 17] $\mathcal{C}_0$ perturbs to a locally invariant slow manifold $\mathcal{C}_\varepsilon$ that is normally hyperbolic attracting as well. In this setting and under sufficiently small noise, a typical sample path of (1.1) starting near $\mathcal{C}_\varepsilon$ is going to fluctuate around $\mathcal{C}_\varepsilon$ and also slowly drifts according to a stochastic perturbation of the slow subsystem, see Figure 1. A fundamental theory to estimate the probability of such a sample path to stay near the slow manifold was developed by Berglund and Gentz [3, 4]. Note that their theory has extensions to bifurcations [2] and global dynamical noisy fast-slow patterns [6].

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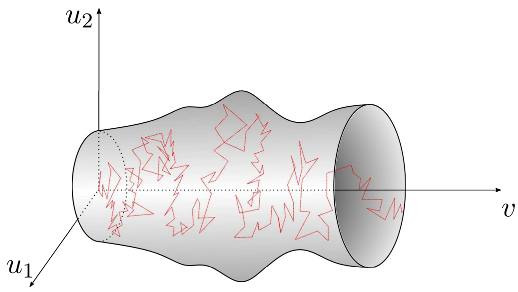

FIGURE 1. Sketch of a sample path (red) inside an ellipsoidal neighbourhood $\mathcal{E}_r$ (grey) for the finite-dimensional SODE setting with two fast and one slow variable. In the SPDE case, we are going to view the u-direction as infinite-dimensional.

It would be very desirable to have a generalization of this theory to fast-slow SPDEs. Examples of such systems arising in applications are the FitzHugh-Nagumo SPDE [7, 13], slowly-driven amplitude/modulation equations [8, 14], and degenerate controlled SPDEs [19, 20]. There are certainly many other important examples as most PDEs arising in applications have parameters, which quite often are *slow variables*, and those PDEs should frequently have *noise terms*, e.g. due to finite-size effects or external forces. An important class of such equations can be formulated as

$$\left\{ \begin{array}{lcl} \mathrm{d}u & = & [Au + f(u,v,\varepsilon)]\ \mathrm{d}t + \sigma\ \mathrm{d}W^Q, \\ \mathrm{d}v & = & \varepsilon g(u,v,\varepsilon)\ \mathrm{d}t, \end{array} \right. \tag{1.3}$$

where A is a differential operator so that $S(t) := \mathrm{e}^{tA}$ is a strongly continuous semigroup on a given (spatial) function space $\mathcal{H}$ and W^Q is a Q-Wiener process. One natural setting occurs when $\mathcal{H}$ is a Hilbert space and we view (1.3) as an evolution equation on $\mathcal{H}$. By requiring $f(0,v,\varepsilon) = 0$ the analogue of $\mathcal{C}_0$ and $\mathcal{C}_\varepsilon$ is just the zero solution $\{u \equiv 0\}$. Then one is charged with providing estimates on the slowly-drifting process (starting say on $\{u \equiv 0\}$) to stay in a suitable neighbourhood of $\{\equiv 0\}$ inside $\mathcal{H}$.

In this paper, as a first step towards the case (1.3), we consider the situation where the system is reduced to a scalar linear non-autonomous SPDE. This is already the theoretical analogue to the key step in the SODE theory of Berglund/Gentz; see the linearized parts of the estimates in [4, Sec. 5.1].

We investigate the SPDE on a bounded interval so that the solution can be expressed by a Fourier series and the SPDE is naturally reformulated as an infinite-dimensional system of SODEs. The linear reaction term consists of a time-dependent coefficient and a non-local operator which generates linear couplings between the first $k_* - 1$ modes. Thus, it is natural to split the system into two parts: the first part consisting of the first $k_* - 1$ coupled low-frequency modes and the second part consisting of infinitely many decoupled high-frequency modes. Both components are estimated by Bernstein-type inequalities and the limiting process in the second part is approached by an iteration argument. The probability to exit a certain neighbourhood around a stable slow manifold of the system (in our case the steady state $u \equiv 0$) can then be estimated by convolving the corresponding probabilities for finite and high frequencies. Systems of the form presented here arise not only directly in numerical spectral Galerkin methods for SPDEs [18] but also in the context of inertial manifolds defined via a finite number of effective Fourier modes; see [27] for the classical deterministic setting.

The remaining part of this paper is structured as follows: In Section 2 we precisely define the setting of the SPDE case and we clarify the approach using finite-dimensional approximations. Our main result and technical contributions are contained in Section 3. Lastly, we provide a summary and an outlook on possible generalizations in Section 4.

2. SPDE SETTING & GALERKIN APPROXIMATION

Let us begin by describing the setting at hand in some detail. For a thorough presentation of mild solution theory for SPDEs we refer the reader to [10].

2.1. **SPDE setting.** We consider the Hilbert space $L^2([0,L])$ for some $L>0$ and a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $Q: L^2([0,L]) \to L^2([0,L])$ be a symmetric, non-negative, bounded, linear operator of trace class and let W^Q be a $L^2([0,L])$-valued Q-Wiener process on $(\Omega,\mathcal{F},\mathbb{P})$. Let $\{e_k\}_k$ denote the eigenfunctions of the covariance operator Q with associated non-negative eigenvalues $\{\lambda_k\}_k$, so that

$$Qe_k = \lambda_k e_k \text{ for all } k. \tag{2.1}$$

The eigenfunctions form a complete orthonormal system in $L^2([0,L])$ and by assumption we have $\sum_k \lambda_k < \infty$. Now, for any $s \in [0,t]$ the random variable $W^Q(s)$ can be represented as

$$W^Q(s) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} W_k(s) e_k, \tag{2.2}$$

where $\{W_k\}_k$ is a sequence of standard $\mathbb{R}$-valued independent Brownian motions on $(\Omega,\mathcal{F},\mathbb{P})$. The series is convergent in $L^2(\Omega,\mathcal{F},\mathbb{P}; L^2([0,L]))$ (cf. [10, Prop. 4.3]).

As mentioned in the introduction, we will focus on a special, yet crucial, case of equation (1.3) throughout the paper. In particular, we choose $A = \frac{\partial^2}{\partial x^2}$, $g \equiv 1$ and $f(u,s,\varepsilon) = a(s)u(s) + Bu(s)$, with $a: \mathbb{R} \to \mathbb{R}$ being bounded and measurable and B being a non-local operator as defined below. After changing to the slow time scale, we end up with the following equation

$$\mathrm{d}u(s) = \frac{1}{\varepsilon}\left[\frac{\partial^2}{\partial x^2}u(s) + a(s)u(s) + Bu(s)\right] \mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}} \mathrm{d}W^Q(s), \tag{2.3}$$

which is interpreted as a linear evolution equation in $L^2([0,L])$. We equip the equation with homogeneous Dirichlet boundary conditions, i.e, we have $u(s,0) = u(s,L) = 0$ for all $s \in [0,t]$. Furthermore, we assume as initial condition $u(0,x) = 0$ for all $x \in [0,L]$.

The domain of the operator $A = \frac{\partial^2}{\partial x^2}$ in $L^2([0,L])$ under homogeneous Dirichlet boundary conditions is given by $D(A) = H^2([0,L]) \cap H_0^1([0,L])$. $D(A)$ is dense in $L^2([0,L])$ and A has a complete orthonormal set of eigenfunctions

$$\{\phi_k\}_k = \left\{x \mapsto \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right)\right\}_k$$

in $L^2([0,L])$ with associated eigenvalues $\mu_k = -\frac{k^2\pi^2}{L^2}$. In particular, A generates a strongly continuous semigroup $S(s) = \mathrm{e}^{sA}$ in $L^2([0,L])$, namely the heat semigroup. B is defined via its action on the eigenfunctions $\{\phi_k\}_k$ in the following way

$$B\phi_k(x) = \begin{cases} \sum_{\ell=1}^{k_*-1} b_k^\ell \phi_\ell(x) & , k \leq k_* - 1, \\ 0 & , k \geq k_*. \end{cases} \tag{2.4}$$

The operator $\mathcal{L}(s) := \frac{1}{\varepsilon}\left[\frac{\partial^2}{\partial x^2} + a(s) + B\right]$ generates a strongly continuous evolution family $(R(s,r))_{0\leq r\leq s\leq t}$ and (2.3) admits a unique mild solution in $L^2([0,L])$ (cf. [28], where a cylindrical Q-Wiener process is considered)

$$u(s) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^s R(s,r) \, \mathrm{d}W^Q(r). \tag{2.5}$$

2.2. **Spectral Galerkin approximation.** In this subsection, we derive a finite-dimensional approximation of (2.3) by using a spectral Galerkin approximation. We are looking for an expansion in terms of the eigenfunctions of the linear operator A and for simplicity we assume $\phi_k = e_k$ for all k. For $m \geq k_*$ consider the m-dimensional space $V_m := \mathrm{span}\{\phi_1, ..., \phi_m\}$ and define the orthogonal projection operator $P_m : L^2([0,L]) \to V_m$ by

$$P_m h = \sum_{k=1}^{m} \hat{h}_k \phi_k, \qquad \hat{h}_k = \int_0^L \phi_k(x) h(x) \, \mathrm{d}x \quad \text{ for all } h \in L^2([0,L]). \tag{2.6}$$

Now, let $u_m(s)$ be the m-dimensional Galerkin approximation of (2.5), i.e., the solution of the projected equation

$$\mathrm{d}u_m = \frac{1}{\varepsilon} P_m \left[\frac{\partial^2}{\partial x^2} u_m + a(s) u_m + B u_m\right] \,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\, P_m \,\mathrm{d}W^Q. \tag{2.7}$$

Then, under suitable assumptions, $u_m(s)$ converges in the L^∞-topology to $u(s)$ as $m \to \infty$ (see [9] for the details regarding this convergence). We further calculate

$$P_m u(s) = \sum_{k=1}^{m} \hat{u}_k(s)\phi_k, \;\; \hat{u}_k(s) := \int_0^L \phi_k(x) u(s,x) \,\mathrm{d}x,$$

and using integration by parts

$$\int_0^L \phi_k(x) \frac{\partial^2}{\partial x^2} u(s,x) \,\mathrm{d}x = \int_0^L \frac{\partial^2}{\partial x^2}\phi_k(x) u(s,x) \,\mathrm{d}x = \mu_k \int_0^L \phi_k(x) u(s,x) \,\mathrm{d}x = \mu_k \hat{u}_k(s),$$

so that we obtain

$$P_m \frac{\partial^2}{\partial x^2} u(s) = \sum_{k=1}^{m} \mu_k \hat{u}_k(s)\phi_k.$$

Similarly, we also project the noise term

$$P_m W^Q(s) = \sum_{k=1}^{m} \sqrt{\lambda_k} W_k(s)\phi_k,$$

since one can simply calculate

$$\int_0^L \phi_k(x) \sum_{j=1}^{\infty} \sqrt{\lambda_j} W_j(s)\phi_j(x) \,\mathrm{d}x = \sum_{j=1}^{\infty} \sqrt{\lambda_j} W_j(s) \int_0^L \phi_k(x)\phi_j(x) \,\mathrm{d}x = \sqrt{\lambda_k} W_k(s).$$

Furthermore, the non-autonomous part of the drift term gives

$$P_m a(s) u(s) = \sum_{k=1}^{m} a(s) \hat{u}_k(s)\phi_k,$$

and

$$P_m B u(s) = \sum_{\ell=1}^{k_*-1} \sum_{k=1}^{k_*-1} b_k^{\ell} \hat{u}_k(s)\phi_\ell$$

In summary, (2.7) is equivalent to the following finite-dimensional system of SODEs

$$\begin{pmatrix} \mathrm{d}U_1(s) \\ \mathrm{d}U_2(s) \end{pmatrix} = \frac{1}{\varepsilon} \underbrace{\left(\begin{array}{c|c} J_1(s) & 0 \\ \hline 0 & J_2(s) \end{array}\right)}_{=:J(s)} \underbrace{\begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}}_{=:U_m(s)} \mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}} \left(\begin{array}{c|c} F_1 & 0 \\ \hline 0 & F_2 \end{array}\right) \begin{pmatrix} \mathrm{d}\mathbf{W}_1(s) \\ \mathrm{d}\mathbf{W}_2(s) \end{pmatrix}, \tag{2.8}$$

where

$$J_1(s) := \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mu_{k_*-1} \end{pmatrix} + a(s)\,\mathrm{id}_{k_*-1} + \underbrace{\begin{pmatrix} b_1^1 & b_2^1 & \dots & b_{k_*-1}^1 \\ b_1^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{k_*-1}^{k_*-2} \\ b_1^{k_*-1} & \dots & b_{k_*-2}^{k_*-1} & b_{k_*-1}^{k_*-1} \end{pmatrix}}_{=:\mathbf{B}}, \tag{2.9}$$

$$J_2(s) := \begin{pmatrix} \mu_{k_*} + a(s) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mu_m + a(s) \end{pmatrix}, \tag{2.10}$$

and $U_1(s) := (\hat{u}_1(s), \dots, \hat{u}_{k_*-1}(s))^\top$, $U_2(s) := (\hat{u}_{k_*}(s), \dots, \hat{u}_m(s))^\top$, $F_1 := \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_{k_*-1}})$, $F_2 := \operatorname{diag}(\sqrt{\lambda_{k_*}}, \dots, \sqrt{\lambda_m})$, $\mathbf{W}_1(s) = (W_1(s), \dots, W_{k_*-1}(s))^\top$, $\mathbf{W}_2(s) = (W_{k_*}(s), \dots, W_m(s))^\top$.
Let the following assumptions hold throughout the rest of the paper:

Assumptions 2.1.

(a) $J_{i,j} \in C^1([0,t], \mathbb{R})$, *for all* $i, j = 1, ..., m$ *and the derivatives are uniformly bounded by a constant* M.
(b) $a_- < a(s) < a_+$ *for all* $s \in [0,t]$ *where* $a_-, a_+ \in \mathbb{R}$.
(c) $\mu_1 + a_+ + \|\mathbf{B}\|_{\text{op}} =: -\overline{\kappa} < 0$.
(d) $\lambda_k \neq 0$ *for all* $k = 1, ..., m$.

The last three assumptions are needed to prove the existence of a non-degenerate neighbourhood within which sample paths are going to stay for long times.

Note that, since Q is a trace class operator there exist $c > 0$ and $p > 3$ such that $\frac{k^2}{\lambda_k} \geq ck^p$. In the following we only assume that $p > 1$ and we expect that the result generalizes to cylindrical Q-Wiener processes which fulfill this growth condition. In particular, this relation between the eigenvalues of J_2 (which grow like $|\mu_k| \sim k^2$) and the eigenvalues of the covariance operator Q (which decrease monotonically to zero as $k \to \infty$) guarantees that the deterministic decay towards zero of the drift term dominates the noisy fluctuations in the higher modes. Furthermore, note that the deterministicaly attracting slow manifold of (2.8) is given by $\mathcal{C}_\varepsilon = \{U_m(s) = 0\}$ for $s \in [0,t]$ since $U_m(s) \equiv 0$ solves the problem without noise for any $\varepsilon = 0$ and any $s \in [0,t]$, and the sign conditions we assumed above guarantee that $\mathcal{C}_\varepsilon$ is attracting.

3. Exponential sample path estimates

3.1. **Main result and proof strategy.** We consider equation (2.3) on the spatial interval $[0, L]$, $L > 0$, and time interval $[0,t]$, where we assume

$$t = \Lambda\varepsilon,$$

together with homogeneous Dirichlet boundary conditions and initial condition $u(0) = 0$, as discussed in Section 2.1. Let u be the mild solution to this problem and let U_m be the m-dimensional Galerkin approximation, as discussed in Section 2.2. Furthermore, let Assumption 2.1 hold. It is helpful to introduce some notation to deal with various constants appearing in our subsequent arguments. Let

$$C_1 := C_1(\gamma) := C\frac{\underline{\kappa} + \beta}{(k_* - 1)^3\sigma^2\lambda_1} \exp\left(2\Lambda(\overline{\kappa} - \underline{\kappa} - 2\beta)\right) \exp\left(-\frac{2\gamma}{\underline{\kappa}}(\underline{\kappa} + \beta)\right), \tag{3.1}$$

where $\gamma > 0$ is chosen arbitrarily, C, β are constants depending on the particular form of $\mathbf{B}$ and $\underline{\kappa}$, $\overline{\kappa}$ are lower and upper bounds on the eigenvalues of $J_1(s)$ (see Section 3.3 for details). Likewise, for arbitrary $\tilde{\gamma} > 0$ we define

$$C_2 = C_2(\tilde{\gamma}) := \frac{c\tilde{c}\exp\left(-2\tilde{\gamma}\right)\pi^2}{\sigma^2 L^2}, \tag{3.2}$$

where $c > 0$ is the constant such that $\frac{k^2}{\lambda_k} \geq ck^p$, and $\tilde{c} > 0$ is chosen such that $|\mu_k + a_+| \geq \tilde{c}|\mu_k|$. We also introduce the notation

$$H_*(k) := \frac{\ln\left(2\left\lceil\frac{\Lambda}{\tilde{\gamma}}|a_- + \mu_k|\right\rceil\right)}{C_2 k^p}, \tag{3.3}$$

and we note that, defining $H_*^m := H_*(k_* + m)$ for $m \in \mathbb{N}$, we have $\sum_{m=0}^\infty H_*^m < \infty$. Furthermore, we define

$$\eta_* := \sum_{m=0}^{\infty} H_*^m + \frac{2p}{(p^2-1)C_2} + 2\delta, \tag{3.4}$$

where δ is chosen such that $\frac{1}{C_2\delta} \geq 1$, and

$$\zeta_* := \frac{\ln\left(2\lceil \Lambda \underline{\kappa}/\gamma \rceil\right)}{C_1}, \tag{3.5}$$

as well as

$$\xi_* := \frac{1}{C_2\delta} \frac{\exp\left(-C_2 k_*^p \delta\right)}{1 - \exp\left(-C_2 p k_*^{p-1}\delta\right)}. \tag{3.6}$$

Our main result then reads as follows:

Theorem 3.1. *Let $\gamma, \tilde{\gamma} > 0$ be arbitrary. For $H \geq \eta_* + \zeta_*$ we have*

$$\begin{aligned} \mathbb{P}\left(\sup_{0\leq s\leq t} \|u(s)\|^2 \geq H\right) &\leq \exp\left(-C_1\left(H - \eta_* - \zeta_*\right)\right) \\ &\quad + \xi_* C_1 \frac{|\exp\left(-C_1(H-\eta_*-\zeta_*)\right) - \exp\left(-C_2 k_*^p (H - \eta_* - \zeta_*)\right)|}{|C_1 - C_2 k_*^p|}, \end{aligned} \tag{3.7}$$

where $\|\cdot\| = \|\cdot\|_{L^2([0,L])}$ and all the constants are defined as above. The case $C_1 = C_2 k_*^p$ is to be understood in the sense of taking the derivative.

Theorem 3.1 tells us the following: For H large enough, the probability that the solution to equation (2.3) deviates more than H from the deterministic slow manifold within an ε-small time interval, is exponentially small in H. The larger p, i.e., the faster the eigenvalues of the covariance operator Q decrease, the smaller is the lower bound on H for which we can guarantee this exponential decay.

Let us briefly outline the strategy of the proof. Note that by Parseval's identity we have for $H > 0$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \|u(s)\|^2 \geq H\right) = \mathbb{P}\left(\sup_{0\leq s\leq t} \sum_{k=1}^{\infty} |\hat{u}_k(s)|^2 \geq H\right). \tag{3.8}$$

For readability we write $u_k(s)$ for $\hat{u}_k(s)$ from now on. The main idea to prove Theorem 3.1 is to split the infinite sum in (3.8) into two parts, one containing the first $k_* - 1$ components and the other one containing the last $m - k_* + 1$ components, where we let m tend to ∞. We call the first sum the finite-frequency part and the second sum the high-frequency part. The two parts can be estimated as follows:

Proposition 3.2. *For arbitrary $\gamma > 0$ we have for $H \geq \zeta_*$*

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \sum_{k=1}^{k_*-1} |u_k(s)|^2 \geq H\right) \leq \exp\left(-C_1(H - \zeta_*)\right). \tag{3.9}$$

Proposition 3.3. *For arbitrary $\tilde{\gamma} > 0$ we have for $H \geq \eta_*$*

$$\mathbb{P}\left(\sum_{k=k_*}^{\infty} \sup_{0\leq s\leq t} |u_k(s)|^2 \geq H\right) \leq \xi_* \exp\left(-C_2 k_*^p (H - \eta_*)\right). \tag{3.10}$$

Proposition 3.2 will be proved in Section 3.3. In order to prove Proposition 3.3 we will use one-dimensional estimates for each component, which we will then combine iteratively, see Section 3.4. Finally, to prove Theorem 3.1 we will concatenate the estimates for the finite-frequency part and the high-frequency part, i.e., Propositions 3.2 and 3.3, see Section 3.5.

Remark 3.4. *Note that similar estimates for one-dimensional and (finite) multi-dimensional SODE systems have been proved in [4]. We use a similar strategy for the proofs, however, in a way which is tailor-made for the linear setting at hand.*

3.2. **Auxiliary results.** Before proving Propositions 3.2, 3.3 and Theorem 3.1 we provide a couple of auxiliary results. The following proposition will become crucial for obtaining estimates on the distribution of the sum of two random variables when exponential estimates for each individual random variable are given.

Proposition 3.5. *Let X, Y be two independent non-negative random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the following two estimates hold*

$$\mathbb{P}(X \geq H) \leq \sum_{i=0}^{n} \xi_X(i) \exp\left(-\kappa_X(i)(H - \eta_X)\right), \qquad \text{for all } H \geq \eta_X, \tag{3.11a}$$

$$\mathbb{P}(Y \geq H) \leq \xi_Y \exp\left(-\kappa_Y(H - \eta_Y)\right), \qquad \text{for all } H \geq \eta_Y, \tag{3.11b}$$

where $n \in \mathbb{N}$, $\xi_X(i), \xi_Y, \kappa_X(i), \kappa_Y > 0$, $\eta_X, \eta_Y \geq 0$ for $i = 0, \ldots, n$. Then, for $H \geq \eta_X + \eta_Y$ we have

$$\begin{aligned} \mathbb{P}(X + Y \geq H) \leq\ & \mathbb{P}(Y \geq H - \eta_X)\left(1 - \sum_{i=0}^{n} \xi_X(i)\right) + \sum_{i=0}^{n} \xi_X(i) \exp\left(-\kappa_X(i)(H - \eta_X - \eta_Y)\right) \\ & - \sum_{i=0}^{n} \xi_Y \xi_X(i) \kappa_X(i) \frac{\exp\left(-\kappa_X(i)(H - \eta_X - \eta_Y)\right) - \exp\left(-\kappa_Y(H - \eta_X - \eta_Y)\right)}{\kappa_X(i) - \kappa_Y}, \end{aligned} \tag{3.12}$$

where the case $\kappa_X(i) = \kappa_Y$ for an $i \in \{1, \ldots, n\}$ is to be understood in the sense of taking the derivative.

Proof. For simplicity we assume $\kappa_X(i) \neq \kappa_Y$ for all $i \in \{1, \ldots, n\}$ in what follows. Further assume $H \geq \eta_X + \eta_Y$. As X and Y are independent we can use the convolution formula for the cumulative distribution function of the sum of two independent random variables. Thus

$$\begin{aligned} &\mathbb{P}(X + Y \geq H) = 1 - \mathbb{P}(X + Y < H) \\ &= 1 - \int_0^H \left(\frac{\mathrm{d}}{\mathrm{d}H_1}(1 - \mathbb{P}(Y \geq H_1))\right)(1 - \mathbb{P}(X \geq H - H_1))\, \mathrm{d}H_1 \\ &= 1 + \int_0^H \frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}(Y \geq H_1)\, \mathrm{d}H_1 - \int_0^H \left(\frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}(Y \geq H_1)\right)\mathbb{P}(X \geq H - H_1)\, \mathrm{d}H_1 \\ &= \mathbb{P}(Y \geq H) - \int_0^H \left(\frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}(Y \geq H_1)\right)\mathbb{P}(X \geq H - H_1)\, \mathrm{d}H_1 \\ &\leq \mathbb{P}(Y \geq H) - \int_0^{H-\eta_X} \left(\frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}(Y \geq H_1)\right)\left[\sum_{i=0}^{n} \xi_X(i) \exp\left(-\kappa_X(i)(H - H_1 - \eta_X)\right)\right] \mathrm{d}H_1 \\ &\quad - \int_{H-\eta_X}^{H} \frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}(Y \geq H_1)\ \mathrm{d}H_1 \\ &= \mathbb{P}(Y \geq H - \eta_X) \underbrace{- \int_0^{H-\eta_X} \left(\frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}(Y \geq H_1)\right)\left[\sum_{i=0}^{n} \xi_X(i) \exp\left(-\kappa_X(i)(H - H_1 - \eta_X)\right)\right] \mathrm{d}H_1}_{=:I_1}, \end{aligned} \tag{3.13}$$

where we have used that $\frac{\mathrm{d}}{\mathrm{d}H_1}\mathbb{P}\left(Y \geq H_1\right) \leq 0$ and (3.11a). With integration by parts and by applying equation (3.11b), we can further estimate I_1

$$
\begin{aligned}
I_1 &= -\left[\mathbb{P}\left(Y \geq H_1\right)\sum_{i=0}^{n}\xi_X(i)\exp\left(-\kappa_X(i)(H-H_1-\eta_X)\right)\right]_{H_1=0}^{H-\eta_X} \\
&\quad + \int_0^{H-\eta_X}\mathbb{P}\left(Y \geq H_1\right)\left[\sum_{i=0}^{n}\xi_X(i)\kappa_X(i)\exp\left(-\kappa_X(i)(H-H_1-\eta_X)\right)\right]\,\mathrm{d}H_1 \\
&\leq -\mathbb{P}\left(Y \geq H-\eta_X\right)\sum_{i=0}^{n}\xi_X(i) + \sum_{i=0}^{n}\xi_X(i)\exp\left(-\kappa_X(i)(H-\eta_X)\right) \\
&\quad + \int_0^{\eta_Y}\sum_{i=0}^{n}\xi_X(i)\kappa_X(i)\exp\left(-\kappa_X(i)(H-H_1-\eta_X)\right)dH_1 \\
&\quad + \underbrace{\int_{\eta_Y}^{H-\eta_X}\xi_Y\exp\left(-\kappa_Y(H_1-\eta_Y)\right)\left[\sum_{i=0}^{n}\xi_X(i)\kappa_X(i)\exp\left(-\kappa_X(i)(H-H_1-\eta_X)\right)\right]\,\mathrm{d}H_1}_{=:I_2} \\
&= -\mathbb{P}\left(Y \geq H-\eta_X\right)\sum_{i=0}^{n}\xi_X(i) + \sum_{i=0}^{n}\xi_X(i)\exp\left(-\kappa_X(i)(H-\eta_X-\eta_Y)\right) + I_2.
\end{aligned}
$$

Now, for I_2 we calculate

$$
\begin{aligned}
I_2 &= \sum_{i=0}^{n}\xi_Y\xi_X(i)\kappa_X(i)\exp\left(\kappa_Y\eta_Y-\kappa_X(i)H+\kappa_X(i)\eta_X\right)\int_{\eta_Y}^{H-\eta_X}\exp\left(H_1(\kappa_X(i)-\kappa_Y)\right)\,\mathrm{d}H_1 \\
&= -\sum_{i=0}^{n}\xi_Y\xi_X(i)\kappa_X(i)\frac{\exp\left(-\kappa_X(i)(H-\eta_X-\eta_Y)\right)-\exp\left(\kappa_Y(H-\eta_X-\eta_Y)\right)}{\kappa_X(i)-\kappa_Y}.
\end{aligned}
$$

Inserting I_2 into I_1 and estimating I_1 as above in equation (3.13) concludes the proof. ☐

We will further need the following results for the iteration step in the high-frequency estimate.

Lemma 3.6. *For $n \in \mathbb{N}$ let $\{x_k\}_{k=0}^n$ be distinct non-negative real numbers. Then*

$$\sum_{i=0}^{n}\prod_{m=0,m\neq i}^{n}\frac{x_m}{x_m-x_i} = 1, \tag{3.14a}$$

$$\sum_{i=0}^{n}\prod_{m=0,m\neq i}^{n}\frac{1}{x_m-x_i} = 0. \tag{3.14b}$$

Proof of (3.14a). Define the auxiliary function $f:\mathbb{R}\to\mathbb{R}$

$$f(x) := \sum_{i=0}^{n}\prod_{m=0,m\neq i}^{n}\frac{1-x/x_m}{1-x_i/x_m}.$$

Then for $k = 0, ..., n$ we have

$$\begin{aligned} f(x_k) &= \sum_{i=0}^{n} \prod_{m=0, m\neq i}^{n} \frac{1 - x_k/x_m}{1 - x_i/x_m} \\ &= \sum_{i=0, i\neq k}^{n} \prod_{m=0, m\neq i}^{n} \frac{1 - x_k/x_m}{1 - x_i/x_m} + \prod_{m=0, m\neq k}^{n} \frac{1 - x_k/x_m}{1 - x_k/x_m} \\ &= \sum_{i=0, i\neq k}^{n} \left(\prod_{m=0, m\neq i, m\neq k}^{n} \frac{1 - x_k/x_m}{1 - x_i/x_m} \right) \underbrace{\frac{1 - x_k/x_k}{1 - x_i/x_p}}_{=0} + 1 \\ &= 1. \end{aligned}$$

Now, $f(x) - 1$ is a polynomial of degree n with $n + 1$ roots, i.e., $f(x) - 1 \equiv 0$. Hence,

$$1 = f(0) = \sum_{i=0}^{n} \prod_{m=0, m\neq i}^{n} \frac{1}{1 - x_i/x_m} = \sum_{i=0}^{n} \prod_{m=0, m\neq i}^{n} \frac{x_m}{x_m - x_i}.$$

Proof of (3.14b). We prove the second identity by induction. For the base case $n = 1$ we have

$$\sum_{i=0}^{n} \prod_{m=0, m\neq i}^{n} \frac{1}{x_m - x_i} = \frac{1}{x_1 - x_0} + \frac{1}{x_0 - x_1} = 0.$$

Now, let (3.14b) hold for arbitrary but fixed $n \in \mathbb{N}$ (inductive hypothesis). Then

$$\begin{aligned} \sum_{i=0}^{n+1} \prod_{m=0, m\neq i}^{n+1} \frac{1}{x_m - x_i} &= \frac{1}{x_{n+1} - x_n} \left[\sum_{i=0}^{n+1} \frac{x_{n+1} - x_i}{\prod_{m=0, m\neq i}^{n+1} (x_m - x_i)} - \sum_{i=0}^{n+1} \frac{x_n - x_i}{\prod_{m=0, m\neq i}^{n+1} (x_m - x_i)} \right] \\ &= \frac{1}{x_{n+1} - x_n} \left[\sum_{i=0}^{n} \frac{1}{\prod_{m=0, m\neq i}^{n} (x_m - x_i)} - \sum_{i=0, i\neq n}^{n+1} \frac{1}{\prod_{m=0, m\neq i, m\neq n}^{n+1} (x_m - x_i)} \right] \\ &= \frac{1}{x_{n+1} - x_n} [0 - 0] = 0, \end{aligned}$$

where we have used the inductive hypothesis in the last line. □

Corollary 3.7. *For $k_* \in \mathbb{N}$, $a, b, c \in \mathbb{N}$ let us define the following quotient*

$$Q_{b,c}^{a} := \frac{(k_* + a)^p}{(k_* + b)^p - (k_* + c)^p}, \tag{3.15}$$

which will appear in the estimate for the high-frequency part. For $n \in \mathbb{N}$ we have

$$\sum_{i=0}^{n} \prod_{n=0, m\neq i}^{n} Q_{m,i}^{m} = 1, \tag{3.16a}$$

$$\sum_{i=0}^{n+1} \prod_{m=0, m\neq i}^{n+1} Q_{m,i}^{0} = 0, \tag{3.16b}$$

$$1 - Q_{i,n+1}^{i} = Q_{n+1,i}^{n+1}, \tag{3.16c}$$

$$\sum_{i=0}^{n} \left(\prod_{m=0, m\neq i}^{n} Q_{m,i}^{m} \right) Q_{i,n+1}^{i} = \prod_{m=0}^{n} Q_{m,n+1}^{m}. \tag{3.16d}$$

Proof. Identities (3.16a) and (3.16b) follow directly from Lemma 3.6. The third identity (3.16c) can be easily verified by direct calculation

$$1 - Q^i_{i,n+1} = \frac{(k_*+i)^p - (k_*+n+1)^p}{(k_*+i)^p - (k_*+n+1)^p} - \frac{(k_*+i)^p}{(k_*+i)^p - (k_*+n+1)^p} = \frac{-(k_*+n+1)^p}{(k_*+i)^p - (k_*+n+1)^p} = Q^{n+1}_{n+1,i}.$$

Likewise, the last identity (3.16d) follows from

$$\begin{aligned}
&\sum_{i=0}^{n} \left(\prod_{m=0, m\neq i}^{n} Q^m_{m,i} \right) Q^i_{i,n+1} \\
&= \prod_{m=0}^{n} (k_*+m)^p \left[-\sum_{i=0}^{n} \prod_{m=0,m\neq i}^{n+1} \frac{1}{(k_*+m)^p - (k_*+i)^p} \right] \\
&= \prod_{m=0}^{n} (k_*+m)^p \left[-\underbrace{\sum_{i=0}^{n+1} \prod_{m=0,m\neq i}^{n+1} \frac{1}{(k_*+m)^p - (k_*+i)^p}}_{=0 \text{ by } (3.16b)} + \prod_{m=0}^{n} \frac{1}{(k_*+m)^p - (k_*+n+1)^p} \right] \\
&= \prod_{m=0}^{n} \frac{(k_*+m)^p}{(k_*+m)^p - (k_*+n+1)^p} = \prod_{m=0}^{n} Q^m_{m,n+1}.
\end{aligned}$$

□

3.3. **Finite-frequency estimate.**

Proof of Proposition 3.2. We begin by estimating the eigenvalues of the matrix $J_1(s)$. Let $\psi(s)$ be an eigenvalue of $J_1(s) = \mathrm{diag}(\mu_1, \ldots, \mu_{k_*-1}) + a(s)\mathrm{id}_{k_*-1} + \mathbf{B}$ with corresponding normed eigenvector w ($\|w\|_2 = 1$), i.e. $J_1(s)w = \psi(s)w$. We have

$$\begin{aligned}
\|\mathbf{B}\|_{\mathrm{op}} &\geq \|\mathbf{B}w\|_2 \\
&= \|\mathrm{diag}(\psi(s) - a(s) - \mu_1, \ldots, \psi(s) - a(s) - \mu_{k_*-1})w\|_2 \\
&\geq \min_{k=1,\ldots,k_*-1} |\psi(s) - a(s) - \mu_k|.
\end{aligned}$$

This estimate yields an upper and a lower bound on $\psi(s)$:

$$\begin{aligned}
\psi(s) &\leq a(s) + \max_{k=1,\ldots,k_*-1} \mu_k + \|\mathbf{B}\|_{\mathrm{op}} \leq a_+ + \mu_1 + \|\mathbf{B}\|_{\mathrm{op}} =: -\overline{\kappa}, \\
\psi(s) &\geq a(s) + \min_{k=1,\ldots,k_*-1} \mu_k - \|\mathbf{B}\|_{\mathrm{op}} \geq a_- + \mu_{k_*-1} - \|\mathbf{B}\|_{\mathrm{op}} =: -\underline{\kappa},
\end{aligned}$$

with $0 < \overline{\kappa} < \underline{\kappa}$ (cf. Assumption 2.1 (c)). Now, let $U(s)$ be the solution to the $k_* - 1$-dimensional system

$$U_1(s) = \frac{1}{\varepsilon} J_1(s) U_1(s) + \frac{\sigma}{\sqrt{\varepsilon}} F_1 \, \mathrm{d}\mathbf{W}_1(s).$$

Using Duhamel's principle the solution can be represented as follows

$$U_1(s) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^s \exp\left(\frac{1}{\varepsilon} \alpha(s,\tau) \right) F_1 \, \mathrm{d}\mathbf{W}_1(\tau),$$

with $\alpha(s,\tau) := \int_\tau^s J_1(r) \, \mathrm{d}r$. Furthermore, define $\alpha(s) := \alpha(s,0)$. Since we have an upper and a lower bound for the eigenvalues of $J_1(s)$, we can obtain the following estimates

$$\left\| \exp\left(\frac{1}{\varepsilon} \alpha(s) \right) \right\|_{\mathrm{op}} \leq \overline{C} \exp\left(-\frac{s}{\varepsilon} (\overline{\kappa} - \beta) \right), \tag{3.17}$$

$$\left(\exp\left(-\frac{1}{\varepsilon}\alpha(\tau)\right)\right)_{i,j} \leq \underline{C}\exp\left(\frac{\tau}{\varepsilon}(\underline{\kappa}+\beta)\right), \tag{3.18}$$

where the constant $\beta \geq 0$ comes from the polynomial part appearing in non-diagonalizable matrices and $\underline{C}, \overline{C}$ are time-independent constants as well.

Let us now introduce a partition of the time interval $[0,t]$ by $0 = s_0 < s_1 < \ldots < s_N = t$ with step size $s_{j+1} - s_j = \frac{\varepsilon\gamma}{\underline{\kappa}}$ and $N = \left\lceil \frac{t\underline{\kappa}}{\varepsilon\gamma} \right\rceil$, for some $\gamma > 0$. By applying the Bernstein inequality and using (3.17), (3.18), we can estimate the probability in question as follows

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq s\leq t}\sum_{k=1}^{k_*-1}|u_k(s)|^2 \geq H\right)\\
&\leq \sum_{j=0}^{N-1}\mathbb{P}\left(\sup_{s_j\leq s\leq s_{j+1}}\left\|\frac{\sigma}{\sqrt{\varepsilon}}\int_0^s \exp\left(\frac{1}{\varepsilon}\alpha(s,\tau)\right)F_1\,\mathrm{d}\mathbf{W}_1(\tau)\right\|_2^2 \geq H\right)\\
&\leq \sum_{j=0}^{N-1}\mathbb{P}\left(\sup_{s_j\leq s\leq s_{j+1}}\frac{(k_*-1)^2\sigma^2}{\varepsilon}\max_{1\leq k\leq k_*-1}\lambda_k\left\|\exp\left(\frac{1}{\varepsilon}\alpha(s)\right)\right\|_{\mathrm{op}}^2\left\|\int_0^s\exp\left(-\frac{1}{\varepsilon}\alpha(\tau)\right)e_k\,\mathrm{d}W_k(\tau)\right\|_2^2 \geq H\right)\\
&\leq \sum_{j=0}^{N-1}\mathbb{P}\left(\sup_{s_j\leq s\leq s_{j+1}}\frac{(k_*-1)^3\sigma^2\lambda_1\overline{C}^2}{\varepsilon\exp\left(\frac{2}{\varepsilon}s(\overline{\kappa}-\beta)\right)}\max_{1\leq k,\ell\leq k_*-1}\left|\int_0^s\left(\exp\left(-\frac{1}{\varepsilon}\alpha(\tau)\right)e_k\right)_\ell\mathrm{d}W_k(\tau)\right|^2 \geq H\right)\\
&\leq \sum_{j=0}^{N-1}\max_{1\leq k,\ell\leq k_*-1}\mathbb{P}\left(\sup_{s_j\leq s\leq s_{j+1}}\left|\int_0^s\left(\exp\left(-\frac{1}{\varepsilon}\alpha(\tau)\right)e_k\right)_\ell\mathrm{d}W_k(\tau)\right|^2 \geq \frac{H\varepsilon\exp\left(\frac{2}{\varepsilon}s_j(\overline{\kappa}-\beta)\right)}{(k_*-1)^3\sigma^2\lambda_1\overline{C}^2}\right)\\
&\leq \sum_{j=0}^{N-1}\max_{1\leq k,\ell\leq k_*-1}2\exp\left(-\frac{H\varepsilon}{(k_*-1)^3\sigma^2\lambda_1\overline{C}^2}\exp\left(\frac{2}{\varepsilon}s_j(\overline{\kappa}-\beta)\right)\frac{1}{2\int_0^{s_{j+1}}\left(\exp\left(-\frac{1}{\varepsilon}\alpha(\tau)\right)e_k\right)_\ell^2\mathrm{d}\tau}\right)\\
&\leq \sum_{j=0}^{N-1}2\exp\left(-C\frac{H\varepsilon}{(k_*-1)^3\sigma^2\lambda_1}\exp\left(\frac{2}{\varepsilon}s_j(\overline{\kappa}-\beta)\right)\frac{1}{2\int_0^{s_{j+1}}\exp\left(\frac{2\tau}{\varepsilon}(\underline{\kappa}+\beta)\right)\mathrm{d}\tau}\right)\\
&\leq \sum_{j=0}^{N-1}2\exp\left(-C\frac{H(\underline{\kappa}+\beta)}{(k_*-1)^3\sigma^2\lambda_1}\exp\left(\frac{2}{\varepsilon}s_j(\overline{\kappa}-\underline{\kappa}-2\beta)\right)\exp\left(-\frac{2\gamma}{\underline{\kappa}}(\underline{\kappa}+\beta)\right)\right)\\
&\leq 2\left\lceil\frac{\Lambda\underline{\kappa}}{\gamma}\right\rceil\exp\left(-C\frac{H(\underline{\kappa}+\beta)}{(k_*-1)^3\sigma^2\lambda_1}\exp\left(2\Lambda(\overline{\kappa}-\underline{\kappa}-2\beta)\right)\exp\left(-\frac{2\gamma}{\underline{\kappa}}(\underline{\kappa}+\beta)\right)\right) = \exp\left(-C_1\left(H-\zeta_*\right)\right),
\end{aligned}$$

where $C = \frac{1}{\underline{C}^2\overline{C}^2}$ and C_1, ζ_* are defined in (3.1) and (3.5). □

3.4. **High-frequency estimate.** To obtain an estimate for the high-frequency part we are going to derive estimates for each component $u_k(s)$ with $k \geq k_*$ and then concatenate them via Proposition 3.5. First note that we have the following estimate for one single mode

Lemma 3.8. *For all $k \geq k_*$ we have*

$$\mathbb{P}\left(\sup_{0\leq s\leq t}|u_k(s)|^2 \geq H\right) \leq \exp\left(-C_2k^p(H-H_*(k))\right), \tag{3.19}$$

where $C_2, H_*(k)$ are defined in (3.2) and (3.3).

Proof. Let $k \geq k_*$, the equation for the k-th component reads

$$\mathrm{d}u_k(s) = \frac{1}{\varepsilon}(\mu_k + a(s))u_k(s)\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\sqrt{\lambda_k}\,\mathrm{d}W_k(s), \tag{3.20}$$

and using Duhamel's principle its solution can be represented as

$$u_k(s) = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^s \exp\left(\frac{\alpha_k(s,\tau)}{\varepsilon}\right) \sqrt{\lambda_k}\ \mathrm{d}W_k(\tau), \tag{3.21}$$

with $\alpha_k(s,\tau) = \int_\tau^s (\mu_k + a(r))\ \mathrm{d}r$. We have the following estimate

$$\begin{aligned}\int_0^s \exp\left(\frac{2\alpha_k(s,\tau)}{\varepsilon}\right)\ \mathrm{d}\tau &\leq \int_0^s \exp\left(\frac{2}{\varepsilon}\int_\tau^s (\mu_k + a_+)\ \mathrm{d}r\right)\ \mathrm{d}\tau = \int_0^s \exp\left(\frac{2}{\varepsilon}(\mu_k + a_+)\tau\right)\ \mathrm{d}\tau \\ &= \frac{\varepsilon}{2}\frac{1}{\mu_k + a_+}\left[\exp\left(\frac{1}{\varepsilon}(\mu_k + a_+)s\right) - 1\right] \leq \frac{\varepsilon}{2|\mu_k + a_+|}.\end{aligned} \tag{3.22}$$

Now fix $\tilde{\gamma} > 0$ and we introduce a k-dependent partition $0 = s_0^k < s_1^k < \ldots < s_N^k = t$ of $[0,t] = [0,\varepsilon\Lambda]$ with $-\alpha_k(s_{j+1}^k, s_j^k) = \varepsilon\tilde{\gamma}$ for $0 \leq j < N_k = \left\lceil \frac{|\alpha_k(t)|}{\varepsilon\tilde{\gamma}} \right\rceil$. Then, using the Bernstein inequality and estimate (3.22), we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq s\leq t} |u_k(s)|^2 \geq H\right) &= \mathbb{P}\left(\sup_{0\leq s\leq t}\left|\frac{\sigma}{\sqrt{\varepsilon}}\int_0^s \exp\left(\frac{\alpha_k(s,\tau)}{\varepsilon}\right)\sqrt{\lambda_k}\ \mathrm{d}W_k(\tau)\right| \geq \sqrt{H}\right) \\ &\leq \sum_{j=0}^{N_k-1} \mathbb{P}\left(\sup_{s_j^k\leq s\leq s_{j+1}^k}\left|\int_0^s \exp\left(-\frac{\alpha_k(\tau)}{\varepsilon}\right)\ \mathrm{d}W_k(\tau)\right| \geq \frac{\sqrt{H\varepsilon}}{\sigma\sqrt{\lambda_k}}\inf_{s_j^k\leq s\leq s_{j+1}^k}\exp\left(-\frac{\alpha_k(s)}{\varepsilon}\right)\right) \\ &\leq \sum_{j=0}^{N_k-1} 2\exp\left(-\frac{H\varepsilon}{\sigma^2\lambda_k}\frac{\inf_{s_j^k\leq s\leq s_{j+1}^k}\exp\left(-2\alpha_k(s)/\varepsilon\right)}{2\int_0^{s_{j+1}^k}\exp\left(-2\alpha_k(\tau)/\varepsilon\right)\ \mathrm{d}\tau}\right) \\ &\leq \sum_{j=0}^{N_k-1} 2\exp\left(-\frac{H\varepsilon}{2\sigma^2\lambda_k}\frac{\exp\left(2\alpha_k(s_{j+1}^k, s_j^k)/\varepsilon\right)}{\int_0^{s_{j+1}^k}\exp\left(2\alpha_k(s_{j+1}^k,\tau)/\varepsilon\right)\ \mathrm{d}\tau}\right) \\ &\leq \sum_{j=0}^{N_k-1} 2\exp\left(-\frac{H\varepsilon}{2\sigma^2\lambda_k}\exp\left(-2\tilde{\gamma}\right)\frac{2|\mu_k + a_+|}{\varepsilon}\right) \\ &\leq 2\left\lceil\frac{|\alpha_k(t)|}{\varepsilon\tilde{\gamma}}\right\rceil\exp\left(-\frac{H}{\sigma^2}\exp\left(-2\tilde{\gamma}\right)\frac{\tilde{c}\pi^2}{L^2}ck^p\right) = \exp\left(-C_2k^p(H - H_*(k))\right)\end{aligned}$$

where C_2 and $H_*(k)$ have been defined in (3.2) and (3.3). ☐

We are now going to prove an estimate on a finite sum of components with index $k \geq k_*$. This will be used to prove Proposition 3.3 by finding a bound independent of the number of addends n.

Proposition 3.9. *Let $n \in \mathbb{N}$. For $H \geq \sum_{m=0}^n H_*^m$ we have*

$$\mathbb{P}\left(\sum_{i=0}^n \sup_{0\leq s\leq t}|u_{k_*+i}(s)|^2 \geq H\right) \leq \sum_{i=0}^n \left[\exp\left(-C_2(k_*+i)^p\left(H - \sum_{m=0}^n H_*^m\right)\right)\prod_{m=0,m\neq i}^n Q_{m,i}^m\right], \tag{3.23}$$

where $Q_{m,i}^m$ has been defined in Corollary 3.7.

Proof. We prove the statement inductively. The base case $n = 0$ directly follows from Lemma 3.8. Now, let (3.23) hold for arbitrary but fixed n (inductive hypothesis). Note that $\sum_{i=0}^n \sup_{0\leq s\leq t}|u_{k_*+i}(s)|^2$ and $\sup_{0\leq s\leq t}|u_{k_*+n+1}(s)|^2$ are independent. Furthermore, by the inductive hypothesis we have for the sum the estimate given in equation (3.23) and for the (k_*+n+1)th component we have by Lemma 3.8

$$\mathbb{P}\left(\sup_{0\leq s\leq t}|u_{k_*+n+1}(s)|^2 \geq H\right) \leq \exp\left(-C_2(k_*+n+1)^p(H - H_*^{n+1})\right). \tag{3.24}$$

Now, applying Proposition 3.5 with $\xi_X(i) = \prod_{m=0,m\neq i}^n Q_{m,i}^m$, $\xi_Y = 1$, $\kappa_X(i) = C_2(k_*+i)^p$, $\kappa_Y = C_2(k_*+n+1)^p$, $\eta_X = \sum_{m=0}^n H_*^m$ and $\eta_Y = H_*^{n+1}$, where $i = 0...n$, yields for $H \geq \sum_{m=0}^n H_*^m + H_*^{n+1} = \sum_{m=0}^{n+1} H_*^m$

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{i=0}^{n+1}\sup_{0\le s\le t}|u_{k_*+i}(s)|^2\ge H\right)=\mathbb{P}\left(\sum_{i=0}^{n}\sup_{0\le s\le t}|u_{k_*+i}(s)|^2+\sup_{0\le s\le t}|u_{k_*+n+1}(s)|^2\ge H\right)\\
&\le \mathbb{P}\left(\sup_{0\le s\le t}|u_{k_*+n+1}(s)|^2\ge H-H_*\right)\underbrace{\left(1-\sum_{i=0}^{n}\prod_{m=0,m\neq i}^{n}Q_{m,i}^m\right)}_{=0\text{ by }(3.16a)}\\
&\quad+\exp\left(-C_2(k_*+n+1)^p\left(H-\sum_{m=0}^{n}H_*^m-H_*^{n+1}\right)\right)\sum_{i=0}^{n}\frac{\prod_{m=0,m\neq i}^{n}Q_{m,i}^mC_2(k_*+i)^p}{C_2(k_*+i)^p-C_2(k_*+n+1)^p}\\
&\quad+\sum_{i=0}^{n}\exp\left(-C_2(k_*+i)^p\left(H-\sum_{m=0}^{n}H_*^m-H_*^{n+1}\right)\right)\left(\prod_{m=0,m\neq i}^{n}Q_{m,i}^m-\frac{\prod_{m=0,m\neq i}^{n}Q_{m,i}^mC_2(k_*+i)^p}{C_2(k_*+i)^p-C_2(k_*+n+1)^p}\right)\\
&=\exp\left(-C_2(k_*+n+1)^p\left(H-\sum_{m=0}^{n+1}H_*^m\right)\right)\underbrace{\sum_{i=0}^{n}\prod_{m=0,m\neq i}^{n}Q_{m,i}^mQ_{i,n+1}^i}_{=\prod_{m=0}^{n}Q_{m,n+1}^m\text{ by }(3.16d)}\\
&\quad+\sum_{i=0}^{n}\exp\left(-C_2(k_*+i)^p\left(H-\sum_{m=0}^{n+1}H_*^m\right)\right)\prod_{m=0,m\neq i}^{n}Q_{m,i}^m\underbrace{\left(1-Q_{i,n+1}^i\right)}_{=Q_{n+1,i}^{n+1}\text{ by }(3.16c)}\\
&=\exp\left(-C_2(k_*+n+1)^p\left(H-\sum_{m=0}^{n+1}H_*^m\right)\right)\prod_{m=0,m\neq n+1}^{n+1}Q_{m,n+1}^m\\
&\quad+\sum_{i=0}^{n}\exp\left(-C_2(k_*+i)^p\left(H-\sum_{m=0}^{n+1}H_*^m\right)\right)\prod_{m=0,m\neq i}^{n+1}Q_{m,i}^m\\
&=\sum_{i=0}^{n+1}\exp\left(-C_2(k_*+i)^p\left(H-\sum_{m=0}^{n+1}H_*^m\right)\right)\prod_{m=0,m\neq i}^{n+1}Q_{m,i}^m,
\end{aligned}
$$

where we have used results of Corollary 3.7. □

Proof of Proposition 3.3. Applying Proposition 3.9 yields

$$
\begin{aligned}
&\mathbb{P}\left(\sum_{i=0}^{n}\sup_{0\le s\le t}|u_{k_*+i}(s)|^2\ge H\right)\\
&\le\exp\left(-C_2k_*^pH\right)\sum_{i=0}^{n}\left[\exp\left(-C_2[(k_*+i)^p-k_*^p]H+C_2(k_*+i)^p\sum_{m=0}^{n}H_*^m\right)\prod_{m=0,m\neq i}^{n}Q_{m,i}^m\right].
\end{aligned}
$$

Note that $\prod_{m=0,m\neq i}^{n}Q_{m,i}^m=\prod_{m=0}^{i-1}Q_{m,i}^m\prod_{m=i+1}^{n}Q_{m,i}^m=(-1)^i\prod_{m=0}^{i-1}Q_{i,m}^m\prod_{m=i+1}^{n}Q_{m,i}^m$. By monotone convergence we have

$$
\mathbb{P}\left(\sum_{k=k_*}^{\infty}\sup_{0\le s\le t}|u_k(s)|^2\ge H\right)\le\exp\left(-C_2k_*^pH\right)\lim_{n\to\infty}\sum_{i=0}^{n}\left[(-1)^is_i^n\right], \tag{3.25}
$$

where

$$
s_i^n=\exp\left(-C_2[(k_*+i)^p-k_*^p]H+C_2(k_*+i)^p\sum_{m=0}^{n}H_*^m\right)\prod_{m=0}^{i-1}Q_{i,m}^m\prod_{m=i+1}^{n}Q_{m,i}^m\ge 0.
$$

In what follows, we derive an upper bound for s_i^n being uniform in n

$$\begin{aligned}\prod_{m=0}^{i-1} Q_{i,m}^m &= \exp\left(-\sum_{m=0}^{i-1} \ln\left(\left(\frac{k_*+i}{k_*+m}\right)^p \left(1-\left(\frac{k_*+m}{k_*+i}\right)^p\right)\right)\right) \\ &\leq \exp\left(\int_0^i \ln\left(\frac{1}{1-\left(\frac{k_*+m}{k_*+i}\right)^p}\right)\,\mathrm{d}m\right) = \exp\left(-(k_*+i)\int_{k_*/(k_*+i)}^1 \ln(1-x^p)\,\mathrm{d}x\right) \\ &\leq \exp\left((k_*+i)\int_{k_*/(k_*+i)}^1 x^p\,\mathrm{d}x\right) = \exp\left(\frac{k_*+i}{p+1} - \frac{k_*^{p+1}}{(p+1)(k_*+i)^p}\right) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\prod_{m=i+1}^{n} Q_{m,i}^m &= \prod_{m=i+1}^{n}\left(1+\frac{(k_*+i)^p}{(k_*+m)^p-(k_*+i)^p}\right) \\ &\leq \exp\left(\sum_{m=1}^{n}\ln\left(1+\left(\frac{k_*+i}{m}\right)^p\right)\right) = \exp\left(\ln\left(1+(k_*+i)^p\right)+\sum_{m=2}^{n}\ln\left(1+\left(\frac{k_*+i}{m}\right)^p\right)\right) \\ &\leq \left(1+(k_*+i)^p\right)\exp\left(\int_1^n \ln\left(1+\left(\frac{k_*+i}{m}\right)^p\right)\,\mathrm{d}m\right) \\ &\leq \left(1+(k_*+i)^p\right)\exp\left((k_*+i)\int_{1/(k_*+i)}^{n/(k_*+i)} \frac{1}{y^p}\,\mathrm{d}y\right) \\ &= \left(1+(k_*+i)^p\right)\exp\left(\frac{1}{1-p}\left(\frac{(k_*+i)^p}{n^{p-1}}-(k_*+i)^p\right)\right) \\ &\leq \left(1+(k_*+i)^p\right)\exp\left(\frac{1}{p-1}(k_*+i)^p\right) \quad \text{for all } n\geq 1. \end{aligned} \tag{3.27}$$

Now, let $\delta > 0$ such that $\frac{1}{C_2\delta} \geq 1$. Then

$$\left(1+(k_*+i)^p\right) \leq \frac{1}{C_2\delta}\exp\left(C_2\delta(k_*+i)^p\right). \tag{3.28}$$

By inserting the estimates (3.26) and (3.27) into s_i^n and applying (3.28) we obtain uniformly in $n \geq 0$

$$\begin{aligned}
s_i^n \leq\ & \exp\left(-C_2[(k_*+i)^p - k_*^p]H + C_2(k_*+i)^p \sum_{m=0}^{\infty} H_*^m\right) \\
& \times \exp\left(\frac{k_*+i}{p+1} - \frac{k_*^{p+1}}{(p+1)(k_*+i)^p}\right)(1+(k_*+i)^p)\exp\left((k_*+i)^p \frac{1}{p-1}\right) \\
\leq\ & \exp\left(-C_2[(k_*+i)^p - k_*^p]H + C_2(k_*+i)^p \sum_{m=0}^{\infty} H_*^m\right) \\
& \times \frac{1}{C_2\delta}\exp\left(C_2\delta(k_*+i)^p\right)\exp\left(\frac{(k_*+i)^p}{p-1} + \frac{k_*+i}{p+1} - \frac{k_*^{p+1}}{(p+1)(k_*+i)^p}\right) \\
\leq\ & \frac{1}{C_2\delta}\exp\left(C_2 k_*^p H\right)\exp\left(-\frac{k_*^{p+1}}{(p+1)(k_*+i)^p}\right) \\
& \times \exp\left(-C_2(k_*+i)^p\left(H - \sum_{m=0}^{\infty} H_*^m - \frac{1}{(p-1)C_2} - \frac{1}{(p+1)C_2} - \delta\right)\right) \\
\leq\ & \frac{1}{C_2\delta}\exp\left(C_2 k_*^p\left(\sum_{m=0}^{\infty} H_*^m + \frac{1}{(p-1)C_2} + \frac{1}{(p+1)C_2} + \delta\right)\right) \\
& \times \exp\left(-C_2 p k_*^{p-1} i\left(H - \sum_{m=0}^{\infty} H_*^m - \frac{1}{(p-1)C_2} - \frac{1}{(p+1)C_2} - \delta\right)\right),
\end{aligned}$$

where we have used $(k_*+i)^p \geq k_*^p + p k_*^{p-1} i$ in the last line. Consequently, we get for $H \geq \eta_*$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=0}^{n} s_i^n \leq\ & \frac{1}{C_2\delta}\exp\left(C_2 k_*^p\left(\sum_{m=0}^{\infty} H_*^m + \frac{1}{(p-1)C_2} + \frac{1}{(p+1)C_2} + \delta\right)\right)\sum_{i=0}^{\infty}\exp\left(-C_2 p k_*^{p-1} i\delta\right) \\
=\ & \frac{1}{C_2\delta}\exp\left(C_2 k_*^p\left(\sum_{m=0}^{\infty} H_*^m + \frac{2p}{(p^2-1)C_2} + \delta\right)\right)\frac{1}{1-\exp\left(-C_2 p k_*^{p-1}\delta\right)} \\
=:\ & \xi_* \exp\left(C_2 k_*^p \eta_*\right),
\end{aligned}$$

where ξ_* and η_* are defined in (3.6) and (3.4). Together with (3.25) this completes the proof. □

3.5. **Combining finite- and high-frequency estimates.**

Proof of Theorem 3.1. As outlined before, we split the sum of the components into the finite- and the high-frequency part and obtain

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq s\leq t}\|u(s)\|^2 \geq H\right) &= \mathbb{P}\left(\sup_{0\leq s\leq t}\sum_{k=1}^{\infty}|u_k(s)|^2 \geq H\right) \\
&\leq \mathbb{P}\left(\sup_{0\leq s\leq t}\sum_{k=1}^{k_*-1}|u_k(s)|^2 + \sup_{0\leq s\leq t}\sum_{k=k_*}^{\infty}|u_k(s)|^2 \geq H\right) \\
&\leq \mathbb{P}\left(\sup_{0\leq s\leq t}\sum_{k=1}^{k_*-1}|u_k(s)|^2 + \sum_{k=k_*}^{\infty}\sup_{0\leq s\leq t}|u_k(s)|^2 \geq H\right). \qquad (3.29)
\end{aligned}$$

Now using Proposition 3.2 and 3.3 we can once more apply Proposition 3.5 with $n = 0$, $\xi_X(0) = 1$, $\xi_Y = \xi_*$, $\kappa_X(0) = C_1$, $\kappa_Y = C_2 k_*^p$, $\eta_X = \zeta_*$, $\eta_Y = \eta_*$, and we obtain for $H \geq \eta_* + \zeta_*$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \|u(s)\|^2 \geq H\right) \leq \exp\left(-C_1\left(H - \eta_* - \zeta_*\right)\right) - \xi_* C_1 \frac{\exp\left(-C_1(H-\eta_*-\zeta_*)\right) - \exp\left(-C_2 k_*^p (H-\eta_*-\zeta_*)\right)}{C_1 - C_2 k_*^p},$$

where the case $C_1 = C_2 k_*^p$ is to be understood in the sense of derivatives. □

4. Summary & Outlook

In our main result, Theorem 3.1, we have established that it is possible in a simplified setting to extend finite-dimensional fast-slow SODE bounds [4] near normally hyperbolic slow manifolds to the infinite-dimensional SPDE setting. In particular we have obtained exponential bounds on the probability to stay near a slow manifold. Our proof has shown that it is possible to naturally extend finite-dimensional results to the SPDE (2.3) using a splitting approach into finitely many ('low') frequency modes as stated in Proposition 3.2 and infinitely many ('high') frequency modes as covered by Proposition 3.3. Furthermore, the key idea is to make use of the growth relation between the eigenvalues coming form the deterministic drift term and the eigenvalues of the covariance operator of the noise. The splitting and the iterative treatment of the high-frequency modes are the key steps in the proof. Those steps could be directly converted into a numerical method. Indeed, just keeping the low-frequency modes corresponds to a Galerkin truncation.

Yet, our approach is only a first step towards providing a detailed theory of multiple time scale SPDEs. There are a few direct possible generalizations. For example, it is evident that the decay in the eigenvalues of the operator A and the spectrum of Q are the key objects, which have to be balanced, to obtain exponential error estimates. Hence, we can allow for more general linear operators A with suitable spectra. Furthermore, for suitably regular multiplicative noise in SPDEs considered on finite time scales near a slow manifold, one may often locally estimate the noise near the slow manifold by additive noise terms.

Another next natural step would be to allow linear couplings between the fast and the slow variable, i.e., systems of the form (where we set $B \equiv 0$ here for notational convenience)

$$\begin{cases} \mathrm{d}u &= \frac{1}{\varepsilon}\left[Au + p_1 u + p_2 v\right]\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W^Q, \\ \mathrm{d}v &= \left[p_3 u + p_4 v\right]\,\mathrm{d}s, \end{cases} \tag{4.1}$$

with parameters $p_1, p_2, p_3, p_4 \in \mathbb{R}$, one obtains in the Galerkin approximation 2×2-blocks along the diagonal

$$\begin{pmatrix} \mathrm{d}\hat{u}_1(s) \\ \mathrm{d}\hat{v}_1(s) \\ \vdots \\ \mathrm{d}\hat{u}_m(s) \\ \mathrm{d}\hat{v}_m(s) \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mu_1 + p_1 & p_2 & 0 & \dots & 0 & 0 & 0 \\ p_3 & p_4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu_m + p_1 & p_2 \\ 0 & 0 & 0 & \dots & 0 & p_3 & p_4 \end{pmatrix} \begin{pmatrix} \hat{u}_1(s) \\ \hat{v}_1(s) \\ \vdots \\ \hat{u}_m(s) \\ \hat{v}_m(s) \end{pmatrix} \mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}} \begin{pmatrix} \sqrt{\lambda_1}\mathrm{d}W_1 \\ 0 \\ \vdots \\ \sqrt{\lambda_m}\mathrm{d}W_m \\ 0 \end{pmatrix}. \tag{4.2}$$

The eigenvalues of this block-structured matrix are easily computed. Under certain assumptions on the eigenvalues and with an iterative scheme similar to the one presented here, we expect to obtain exponential bounds on the sample paths as well. In addition, it is natural to conjecture that suitable regular perturbations of order $\mathcal{O}(\varepsilon)$ of the coefficients are not going to alter the results presented here.

However, there are also several extensions, which are substantially more technical. Dealing with nonlinear terms, introducing a general slow SODE including nonlinear terms for g and G, as well as patching shorter time intervals to time intervals of order $\mathcal{O}(1)$ are well-studied for SODEs in [4]. To lift these SODE results to SPDEs is going to require a careful and detailed analysis in future work.

One might also ask, which other possible ways may exist to prove results about concentration of sample paths for fast-slow SPDEs. If we drop the idea to aim for a natural generalizations of the SODE setting, we conjecture that it could be possible to deal with the high-frequency modes more implicitly using Fernique's Theorem [10]. This would provide less explicit estimates, yet potentially yield a shorter proof. Hence, it would be a classical complementary alternative to our iterative scheme. In addition, one could also think about different function space settings such as weighted Sobolev spaces [5], more general abstract Banach spaces [23], or even lifting the results immediately to the functional setting of regularity structures [15]. Yet, these generalizations are far beyond our current setting and are likely to remain challenging long-term open problems.

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Center for Mathematics, Technical University of Munich, Boltzmannstr. 3, 85747 Garching b. Munich, Germany

E-mail address: gnann@ma.tum.de, ckuehn@ma.tum.de, pein@ma.tum.de